

《二次函数综合》课后练习题-参考答案

1. 【解析】(1) $\because$ 抛物线与 x 轴交于点 $A(-1,0)$, 点 $B(-3,0)$

$\therefore$ 设交点式 $y = a(x+1)(x+3)$

$\because OC = OB = 3$, 点 C 在 y 轴负半轴

$\therefore C(0,-3)$

把点 C 代入抛物线解析式得: $3a = -3$

$\therefore a = -1$

$\therefore$ 抛物线解析式为 $y = -(x+1)(x+3) = -x^2 - 4x - 3$

(2) 如图 1, 过点 A 作 $AG \perp BC$ 于点 G , 过点 P 作 $PH \perp x$ 轴于点 H

$\therefore \angle AGB = \angle AGC = \angle PHO = 90^\circ$

$\because \angle ACB = \angle POB$

$\therefore \triangle ACG \sim \triangle POH$

$$\therefore \frac{AG}{PH} = \frac{CG}{OH}$$

$$\therefore \frac{AG}{CG} = \frac{PH}{OH}$$

$\because OB = OC = 3$, $\angle BOC = 90^\circ$

$\therefore \angle ABC = 45^\circ$, $BC = \sqrt{OB^2 + OC^2} = 3\sqrt{2}$

$\therefore \triangle ABG$ 是等腰直角三角形

$$\therefore AG = BG = \frac{\sqrt{2}}{2} AB = \sqrt{2}$$

$$\therefore CG = BC - BG = 3\sqrt{2} - \sqrt{2} = 2\sqrt{2}$$

$$\therefore \frac{PH}{OH} = \frac{AG}{CG} = \frac{1}{2}$$

$$\therefore OH = 2PH$$

设 $P(p, -p^2 - 4p - 3)$

① 当 $p < -3$ 或 $-1 < p < 0$ 时, 点 P 在点 B 左侧或在 AC 之间, 横纵坐标均为负数

$$\therefore OH = -p, PH = -(-p^2 - 4p - 3) = p^2 + 4p + 3$$

$$\therefore -p = 2(p^2 + 4p + 3)$$

$$\text{解得: } p_1 = \frac{-9 - \sqrt{33}}{4}, p_2 = \frac{-9 + \sqrt{33}}{4}$$

$$\therefore P\left(\frac{-9 - \sqrt{33}}{4}, \frac{-9 - \sqrt{33}}{8}\right) \text{ 或 } \left(\frac{-9 + \sqrt{33}}{4}, \frac{-9 + \sqrt{33}}{8}\right)$$

② 当 $-3 < p < -1$ 或 $p > 0$ 时, 点 P 在 AB 之间或在点 C 右侧, 横纵坐标异号

$$\therefore p = 2(p^2 + 4p + 3)$$

$$\text{解得: } p_1 = -2, p_2 = -\frac{3}{2}$$

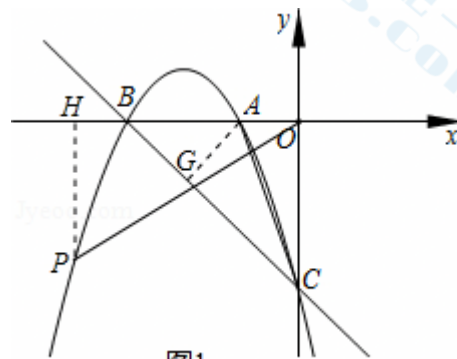


图1

$$\therefore P(-2,1) \text{ 或 } (-\frac{3}{2}, \frac{3}{4})$$

综上所述, 点 P 的坐标为 $(\frac{-9-\sqrt{33}}{4}, \frac{-9-\sqrt{33}}{8})$ 、 $(\frac{-9+\sqrt{33}}{4}, \frac{-9+\sqrt{33}}{8})$ 、 $(-2,1)$ 或 $(-\frac{3}{2}, \frac{3}{4})$.

$$(3) \text{ ①如图 2, } \because x=m+4 \text{ 时, } y=-(m+4)^2-4(m+4)-3=-m^2-12m-35$$

$$\therefore M(m, -m^2-4m-3), N(m+4, -m^2-12m-35)$$

设直线 MN 解析式为 $y=kx+n$

$$\therefore \begin{cases} km+n=-m^2-4m-3 \\ k(m+4)+n=-m^2-12m-35 \end{cases} \quad \text{解得: } \begin{cases} k=-2m-8 \\ n=m^2+4m-3 \end{cases}$$

$$\therefore \text{直线 } MN: y=(-2m-8)x+m^2+4m-3$$

$$\text{设 } D(d, -d^2-4d-3)(m < d < m+4)$$

$\because DE \parallel y$ 轴

$$\therefore x_E = x_D = d, E(d, (-2m-8)d+m^2+4m-3)$$

$$\therefore DE = -d^2-4d-3 - [(-2m-8)d+m^2+4m-3] = -d^2+(2m+4)d-m^2-4m = -[d-(m+2)]^2+4$$

$\therefore$ 当 $d=m+2$ 时, DE 的最大值为 4.

②如图 3, $\because D, F$ 关于点 E 对称,

$$\therefore DE = EF$$

$\therefore$ 四边形 $MDNF$ 是矩形

$\therefore MN = DF$, 且 MN 与 DF 互相平分

$$\therefore DE = \frac{1}{2}MN, E \text{ 为 } MN \text{ 中点}$$

$$\therefore x_D = x_E = \frac{m+m+4}{2} = m+2$$

由①得当 $d=m+2$ 时, $DE=4$

$$\therefore MN = 2DE = 8$$

$$\therefore (m+4-m)^2 + [-m^2-12m-35 - (-m^2-4m-3)]^2 = 8^2$$

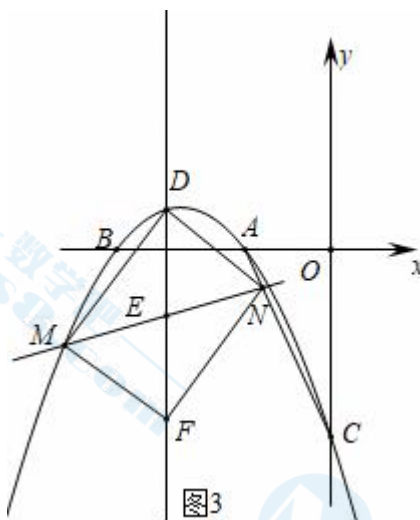
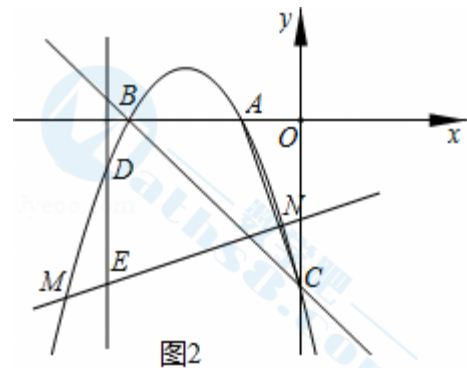
$$\text{解得: } m_1 = -4 - \frac{\sqrt{3}}{2}, m_2 = -4 + \frac{\sqrt{3}}{2}$$

$\therefore m$ 的值为 $-4 - \frac{\sqrt{3}}{2}$ 或 $-4 + \frac{\sqrt{3}}{2}$ 时, 四边形 $MDNF$ 为矩形.

2. 【解析】(1) $\because$ 抛物线 $y=ax^2+bx+c$ 经过点 $A(-4,0)$, $B(2,0)$, $C(0,6)$,

$$\therefore \begin{cases} 16a-4b+c=0 \\ 4a+2b+c=0 \\ c=6 \end{cases} \quad \text{解得} \begin{cases} a=-\frac{3}{4} \\ b=-\frac{3}{2} \\ c=6 \end{cases}, \therefore \text{二次函数的表达式为 } y=-\frac{3}{4}x^2-\frac{3}{2}x+6.$$

(2) 设直线 AE 的表达式为 $y=kx-2$, 则 $0=-4k-2$, 解得 $k=-\frac{1}{2}$,



∴ 直线 AE 的表达式为 $y = -\frac{1}{2}x - 2$,

如图 1, 过点 D 作 $DN \perp x$ 轴于点 G, 交 AE 于点 F,

设 $D(x, -\frac{3}{4}x^2 - \frac{3}{2}x + 6)$, 则 $F(x, -\frac{1}{2}x - 2)$,

$$\therefore DF = -\frac{3}{4}x^2 - \frac{3}{2}x + 6 - (-\frac{1}{2}x - 2) = -\frac{3}{4}x^2 - x + 8,$$

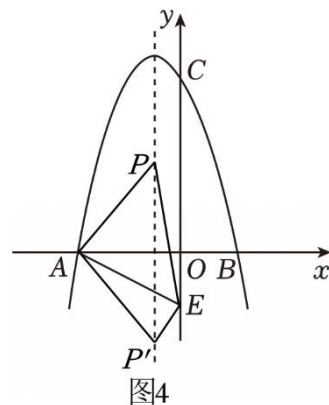
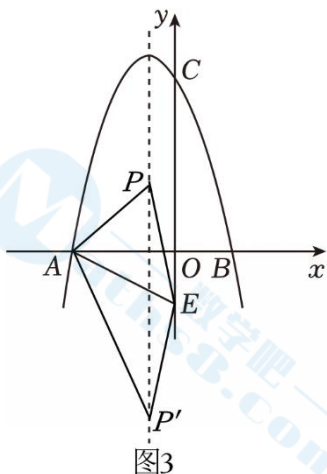
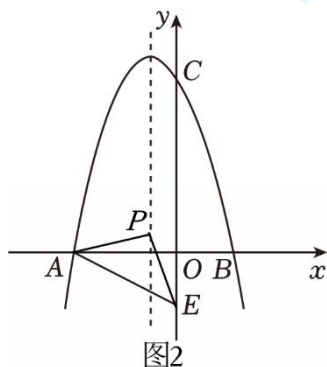
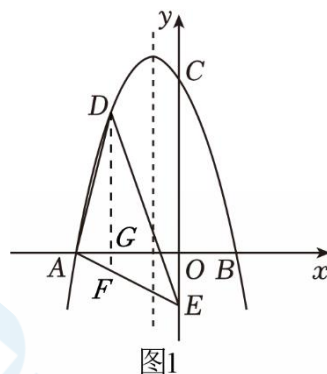
$$\therefore S_{\triangle ADE} = S_{\triangle ADF} + S_{\triangle EDF} = \frac{1}{2}DF \cdot AG + \frac{1}{2}DF \cdot OG = \frac{1}{2} \times 4DF = 2DF,$$

$$\therefore S_{\triangle ADE} = 2(-\frac{3}{4}x^2 - x + 8) = -\frac{3}{2}x^2 - 2x + 16 = -\frac{3}{2}(x + \frac{2}{3})^2 + \frac{50}{3},$$

$$\therefore \text{当 } x = -\frac{2}{3} \text{ 时, } S_{\triangle ADE \text{ 最大}} = \frac{50}{3},$$

$$\therefore \triangle ADE \text{ 面积的最大值是 } \frac{50}{3}.$$

(3)



$$\therefore y = -\frac{3}{4}x^2 - \frac{3}{2}x + 6 = -\frac{3}{4}(x+1)^2 + \frac{27}{4}, \therefore \text{抛物线的对称轴为直线 } x = -1,$$

设 $P(-1, t)$, 如图 2, $\triangle AEP$ 为等腰三角形, 且以 AE 为底边,

$$\therefore PA = PE, \therefore PA^2 = PE^2, \therefore (-1+4)^2 + t^2 = 1^2 + (t+2)^2, \text{解得 } t = 1, \therefore P(-1, 1);$$

如图 3, $\triangle AEP$ 为等腰三角形, 且以 AP 为底边,

$$\therefore AE = PE, \therefore AE^2 = PE^2, \therefore 4^2 + 2^2 = 1^2 + (t+2)^2,$$

$$\text{解得 } t_1 = -2 + \sqrt{19}, t_2 = -2 - \sqrt{19},$$

$$\therefore P(-1, -2 + \sqrt{19}), P'(-1, -2 - \sqrt{19});$$

如图 4, $\triangle AEP$ 为等腰三角形, 且以 PE 为底边,

$$\therefore AE = AP, \therefore AE^2 = AP^2, \therefore 4^2 + 2^2 = (-1+4)^2 + t^2, \text{解得 } t_1 = \sqrt{11}, t_2 = -\sqrt{11},$$

$$\therefore P(-1, \sqrt{11}), P'(-1, -\sqrt{11}),$$

综上所述, 点 P 的坐标为 $(-1, 1)$ 或 $(-1, -2 + \sqrt{19})$ 或 $(-1, -2 - \sqrt{19})$ 或 $(-1, \sqrt{11})$ 或 $(-1, -\sqrt{11})$.

3. 【解析】(1) 抛物线 $y = ax^2 + bx + c (a \neq 0)$ 经过点 $A(3, 0)$, $B(-1, 0)$, $C(0, -3)$. 将点 A , 点 B , 点 C 的坐标分别代入得:

$$\begin{cases} 9a + 3b + c = 0 \\ a - b + c = 0 \\ c = -3 \end{cases}, \text{解得: } \begin{cases} a = 1 \\ b = -2 \\ c = -3 \end{cases}, \therefore \text{抛物线解析式为 } y = x^2 - 2x - 3;$$

(2) 如图, 过点 M 作 $MH \parallel y$ 轴交 AC 于点 H ,

设直线 AC 的解析式为 $y = kx + b$, 将点 A , 点 C 的坐标分别代入得:

$$\begin{cases} 3k + b = 0 \\ b = -3 \end{cases}, \text{解得: } \begin{cases} k = 1 \\ b = -3 \end{cases}, \therefore \text{直线 } AC \text{ 的解析式为 } y = x - 3,$$

$\therefore$ 点 M 的横坐标为 $t (0 < t < 3)$, $\therefore M(t, t^2 - 2t - 3)$, $\therefore H(t, t - 3)$,

$$\therefore HM = t - 3 - (t^2 - 2t - 3) = -t^2 + 3t,$$

$$\therefore S_{\triangle MCA} = S_{\triangle MHC} + S_{\triangle MHA},$$

$$\therefore S_{\triangle MCA} = \frac{1}{2}MH \times (x_M - x_O) + \frac{1}{2}MH \times (x_A - x_M) = \frac{1}{2}MH(x_A - x_O) = \frac{1}{2}MH \times AO = -\frac{3}{2}t^2 + \frac{9}{2}t,$$

$$\text{配方得 } S_{\triangle MCA} = -\frac{3}{2}(t - \frac{3}{2})^2 + \frac{27}{8},$$

$$\therefore -\frac{3}{2} < 0, \therefore \text{当 } t = \frac{3}{2} \text{ 时, 面积取得最大值, 此时 } M(\frac{3}{2}, -\frac{15}{4});$$

(3) 存在以点 B, C, Q, P 为顶点的四边形是平行四边形; 理由如下:

设 $Q(m, 0)$, $P(n, n^2 - 2n - 3)$, $B(-1, 0)$, $C(0, -3)$,

当以 BC, PQ 为对角线时, 由平行四边形的性质得:

$$\begin{cases} -1 + 0 = m + n \\ 0 + (-3) = n^2 - 2n - 3 + 0 \end{cases}, \text{解得: } \begin{cases} m = -1 \\ n = 0 \end{cases} \text{ (不合题意, 舍去) 或 } \begin{cases} m = -3 \\ n = 2 \end{cases},$$

$$\therefore 2^2 - 2 \times 2 - 3 = -3, \therefore P(2, -3);$$

当以 BP, CQ 为对角线时, 由平行四边形的性质得:

$$\begin{cases} -1 + n = m + 0 \\ 0 + (-3) = n^2 - 2n - 3 + 0 \end{cases}, \text{解得: } \begin{cases} m = -1 \\ n = 0 \end{cases} \text{ (不合题意, 舍去) 或 } \begin{cases} m = 1 \\ n = 2 \end{cases},$$

$$\therefore 2^2 - 2 \times 2 - 3 = -3 \therefore P(2, -3);$$

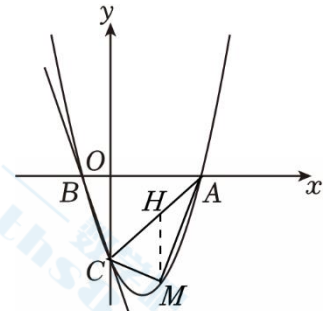
当以 BQ, CP 为对角线时, 由平行四边形的性质得:

$$\begin{cases} -1 + m = n + 0 \\ 0 + 0 = n^2 - 2n - 3 + (-3) \end{cases}, \text{解得: } \begin{cases} m = 2 - \sqrt{7} \\ n = 1 - \sqrt{7} \end{cases} \text{ 或 } \begin{cases} m = 2 + \sqrt{7} \\ n = 1 + \sqrt{7} \end{cases},$$

$$\text{当 } n = 1 - \sqrt{7} \text{ 时, 得: } n^2 - 2n - 3 = 3, \text{ 即 } P(1 - \sqrt{7}, 3)$$

$$\text{当 } n = 1 + \sqrt{7} \text{ 时, 得: } n^2 - 2n - 3 = 3, \text{ 即 } P(1 + \sqrt{7}, 3);$$

综上所述, 存在以点 B, C, Q, P 为顶点的四边形是平行四边形; 此时 P 点坐标为 $(1 - \sqrt{7}, 3)$ 或 $(1 + \sqrt{7}, 3)$ 或 $(2, -3)$.



4. 【解析】(1) 由题意得: $y = a(x+3)(x-1) = a(x^2 + 2x - 3) = ax^2 + bx + 3$, 则 $a = -1$,

故抛物线的表达式为: $y = -x^2 - 2x + 3$;

(2) 由抛物线的表达式知, $OA = OC = 3$, 即 $\triangle ACO$ 为等腰直角三角形,

$\therefore$ 以点 P 、 Q 、 N 为顶点的三角形与 $\triangle AOC$ 相似, 则 $\triangle PNQ$ 为等腰直角三角形,

由点 A 、 C 的坐标知, 直线 AC 的表达式为: $y = x + 3$, 当 $x = -1$ 时, $y = 2$, 即点 $N(-1, 2)$,

$\therefore \angle NAO = 45^\circ$, 故当 $\angle NPQ$ 和 $\angle QPN$ 为直角时, 点 Q 和点 A 重合, 不符合题意;

当 $\angle PNQ$ 为直角时, 则 $y_P = y_N = 2$, 即 $2 = -x^2 - 2x + 3$,

则 $x = -1 - \sqrt{2}$,

则点 $P(-1 - \sqrt{2}, 2)$;

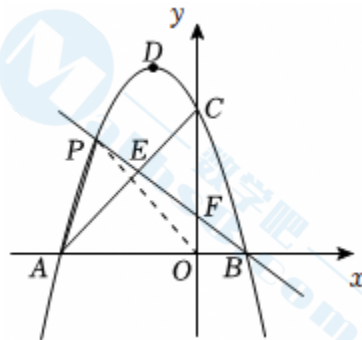
(3) 连接 OP , 设点 $P(m, -m^2 - 2m + 3)$,

由点 P 、 B 的坐标得, 直线 PB 的表达式为: $y = -(m+3)(x-1)$, 则点 $F(0, m+3)$, 即 $PF = m+3$,

则 $k = S_{\triangle APE} - S_{\triangle CEF} = S_{\text{四边形}APFO} - S_{\triangle ACO} = S_{\triangle AOP} + S_{\triangle FPO} - S_{\triangle ACO}$

$$= \frac{1}{2} \times 3 \times (-m^2 - 2m + 3) + \frac{1}{2} (-m)(m+3) - \frac{1}{2} \times 3 \times 3 = -2(m + \frac{9}{8})^2 + \frac{81}{32},$$

即 k 的最大值 $\frac{81}{32}$.



5. 【解析】(1) 将点 $A(-2, 0)$, $B(4, 0)$, 代入 $y = ax^2 + bx + 4$ 得:

$$\begin{cases} 4a - 2b + 4 = 0 \\ 16a + 4b + 4 = 0 \end{cases}, \text{解得: } \begin{cases} a = -\frac{1}{2} \\ b = 1 \end{cases}, \therefore \text{抛物线解析式为 } y = -\frac{1}{2}x^2 + x + 4;$$

(2) $\therefore$ 点 $A(-2, 0)$, $B(4, 0)$, $\therefore$ 抛物线的对称轴为直线 $l: x = \frac{-2+4}{2} = 1$,

设直线 l 与 x 轴交于点 G , 过点 E 作 $ED \perp l$ 于点 D ,

当 F 在 x 轴上方时, 如图:

$\therefore$ 以 B 、 E 、 F 为顶点的三角形是等腰直角三角形, 且 $\angle BFE = 90^\circ$,

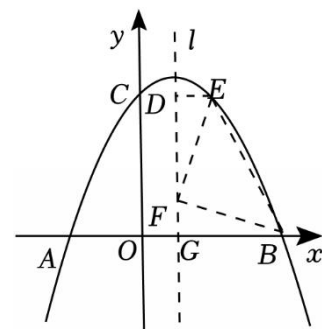
$\therefore EF = BF$,

$\therefore \angle DFE = 90^\circ - \angle BFG = \angle GBF$, $\angle EDF = \angle BGF = 90^\circ$,

$\therefore \triangle DFE \cong \triangle GBF (AAS)$, $\therefore GF = DE$, $GB = FD$,

设 $F(1, m)$, 则 $DE = m$, $DG = DF + FG = GB + FG = 3 + m$,

$\therefore E(1+m, 3+m)$,



$\because E$ 点在抛物线 $y = -\frac{1}{2}x^2 + x + 4$ 上,

$$\therefore 3+m = -\frac{1}{2}(1+m)^2 + (1+m) + 4,$$

解得: $m = -3$ (舍去) 或 $m = 1$, $\therefore F(1, 1)$;

当 F 在 x 轴下方时, 如图:

同理可得 $\triangle DFE \cong \triangle GBF$ (AAS), $GF = DE$, $GB = FD$,

设 $F(1, n)$, 则 $E(1-n, n-3)$,

把 $E(1-n, n-3)$ 代入 $y = -\frac{1}{2}x^2 + x + 4$ 得:

$$n-3 = -\frac{1}{2}(1-n)^2 + (1-n) + 4,$$

解得 $n = 3$ (舍去) 或 $n = -5$, $\therefore F(1, -5)$;

当 E 点与 A 点重合时, 如图所示,

$\because AB = 6$, $\triangle ABF$ 是等腰直角三角形, 且 $\angle BFE = 90^\circ$,

$$\therefore GF = \frac{1}{2}AB = 3, \text{ 此时 } F(1, -3),$$

由对称性可得, 点 $F'(1, 3)$ 也满足条件,

综上所述, $F(1, 1)$ 或 $(1, -5)$ 或 $(1, -3)$ 或 $(1, 3)$;

(3) $OM + \frac{1}{2}ON$ 为定值 6, 理由如下:

设 $P(s, t)$, 直线 AP 的解析式为 $y = dx + f$, BP 的解析式为 $y = gx + h$,

$\because$ 点 $A(-2, 0)$, $B(4, 0)$, $P(s, t)$,

$$\therefore \begin{cases} -2d + f = 0 \\ sd + f = t \end{cases}, \begin{cases} 4g + h = 0 \\ sg + h = t \end{cases}, \text{ 解得: } \begin{cases} d = \frac{t}{s+2} \\ f = \frac{2t}{s+2} \end{cases}, \begin{cases} g = \frac{t}{s-4} \\ h = \frac{4t}{4-s} \end{cases},$$

$\therefore$ 直线 AP 的解析式为 $y = \frac{t}{s+2}x + \frac{2t}{s+2}$, BP 的解析式为 $y = \frac{t}{s-4}x + \frac{4t}{4-s}$,

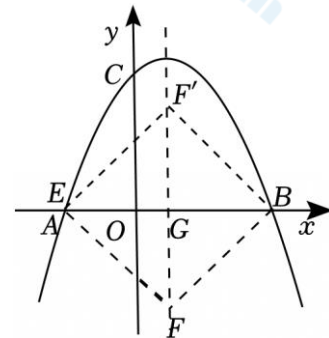
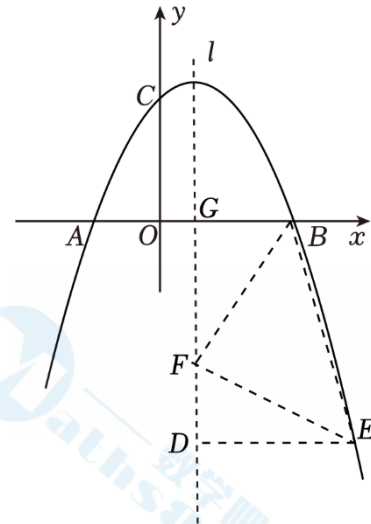
在 $y = \frac{t}{s+2}x + \frac{2t}{s+2}$ 中, 令 $x = 0$ 得 $y = \frac{2t}{s+2}$, $\therefore M(0, \frac{2t}{s+2})$,

在 $y = \frac{t}{s-4}x + \frac{4t}{4-s}$ 中, 令 $x = 0$ 得 $y = \frac{4t}{4-s}$, $\therefore N(0, \frac{4t}{4-s})$,

$\because P(s, t)$ 在抛物线上, $\therefore t = -\frac{1}{2}s^2 + s + 4 = -\frac{1}{2}(s-4)(s+2)$,

$$\therefore OM + \frac{1}{2}ON = \frac{2t}{s+2} + \frac{1}{2} \times \frac{4t}{4-s} = \frac{12t}{-s^2 + 2s + 8} = \frac{-6(s-4)(s+2)}{-(s-4)(s+2)} = 6,$$

$\therefore OM + \frac{1}{2}ON$ 为定值 6.



$$\therefore k_3 m + b_3 = 2, \text{ 即 } b_3 = 2 - k_3 m,$$

$$\text{由 } \begin{cases} y = x^2 + (1-m)x - m = k_3 x + 2 - k_3 m, \\ y = k_3 x + 2 - k_3 m \end{cases},$$

$$\therefore x^2 + (1-m)x - m = k_3 x + 2 - k_3 m,$$

$$\therefore x^2 + (1-m-k_3)x - m - 2 + k_3 m = 0,$$

$$\therefore x_P + x_Q = m + k_3 - 1, \quad x_P \cdot x_Q = -m - 2 + k_3 m,$$

$$\therefore m + k_1 + m + k_2 = m + k_3 - 1, \quad (m + k_1)(m + k_2) = -m - 2 + k_3 m,$$

$$\therefore k_1 + k_2 = -m + k_3 - 1, \quad m^2 + (-m + k_3 - 1)m + k_1 k_2 = -m - 2 + k_3 m,$$

$$\therefore k_1 k_2 = -2, \text{ 即 } OM \cdot ON = -k_1 k_2 = 2.$$

7. 【解析】(1) 由题知 $C_2: y = (x-1)^2 - 4$, $\therefore y = x^2 - 2x - 3$;

(2) 由(1)得 $y = x^2 - 2x - 3$;

令 $x = 0$, 得 $y = -3$,

令 $y = 0$, 得 $x^2 - 2x - 3 = 0$, 解得 $x = -1$ 或 $x = 3$,

$\therefore A$ 点坐标是 $(-1, 0)$, C 点坐标是 $(0, -3)$, B 点坐标是 $(3, 0)$.

设点 E 的坐标是 $(m, m^2 - 2m - 3)$, $\therefore$ 四边形 $ACEF$ 是平行四边形,

$\therefore AC \parallel EF$, $AC = EF$,

$$\therefore \begin{cases} x_C - x_A = x_E - x_F, \\ y_C - y_A = y_E - y_F \end{cases}, \therefore \text{点 } F \text{ 的坐标是 } (m-1, m^2-2m),$$

$$\therefore S_{\triangle AFB} = \frac{1}{2} \times AB \times (-y_F) = \frac{1}{2} \times 4(2m - m^2) = 4m - 2m^2 = \frac{48}{25},$$

$$\text{解得: } m_1 = \frac{4}{5}, \quad m_2 = \frac{6}{5}, \text{ 当 } m_1 = \frac{4}{5} \text{ 时, } x_F = -\frac{1}{5} < 0,$$

不符合点 F 在第四象限的条件, 故舍去,

$$\therefore E \text{ 点的横坐标为 } \frac{6}{5}.$$

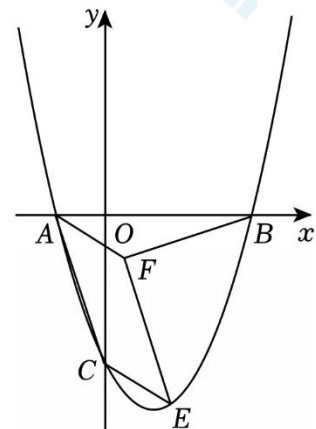


图1

(3) 联立直线 $y_{MN} = k_1 x$ 与抛物线 $y = x^2 - 2x - 3$,

$$\text{得到: } x^2 - (2+k_1)x - 3 = 0, \therefore x_M x_N = -3,$$

同理可得: $x_G x_H = -3$.

设直线 MG 的解析式是 $y_{MG} = k_3 x + b_3$,

直线 HN 的解析式是 $y_{HN} = k_4 x + b_4$,

联立直线 $y_{MG} = k_3 x + b_3$ 与抛物线 $y = x^2 - 2x - 3$,

$$\text{得到: } x^2 - (2+k_3)x - (3+b_3) = 0,$$

$$\therefore x_M + x_G = 2 + k_3, \quad x_M x_G = -3 - b_3,$$

同理可得: $x_H + x_N = 2 + k_4, \quad x_H x_N = -3 - b_4$,

$$\text{又令 } k_3 x + b_3 = 0 \text{ 得: } x_P = -\frac{b_3}{k_3},$$

$$\text{同理 } x_Q = -\frac{b_4}{k_4},$$

$$\therefore \frac{1}{OP} - \frac{1}{OQ} = \frac{k_3}{b_3} + \frac{k_4}{b_4};$$

$$\therefore k_3 = x_M + x_G - 2, \quad b_3 = -x_M x_G - 3,$$

$$k_4 = x_H + x_N - 2 = \frac{-3}{x_G} + \frac{-3}{x_M} - 2 = \frac{-3x_M - 3x_G - 2x_M x_G}{x_M x_G},$$

$$b_4 = -x_H x_N - 3 = \frac{-3}{x_G} \cdot \frac{-3}{x_M} - 3 = \frac{-9 - 3x_G x_M}{x_G x_M},$$

$$\begin{aligned} \therefore \frac{1}{OP} - \frac{1}{OQ} &= \frac{k_3}{b_3} + \frac{k_4}{b_4} = \frac{x_M + x_G - 2}{-x_M x_G - 3} + \frac{\frac{-3x_M - 3x_G - 2x_M x_G}{x_M x_G}}{\frac{-9 - 3x_G x_M}{x_G x_M}} \\ &= \frac{x_M + x_G - 2}{-x_M x_G - 3} + \frac{-3x_M - 3x_G - 2x_M x_G}{-9 - 3x_G x_M} \\ &= \frac{-6 - 2x_M x_G}{-9 - 3x_G x_M} = \frac{2}{3}. \end{aligned}$$

8. 【解析】(1) 设直线 AB 与 y 轴的交点为点 E , 如图 1,

$$\therefore AB \parallel x \text{ 轴}, \quad AB = 2, \quad \therefore AE = BE = 1,$$

$$\therefore \angle AOB = 90^\circ, \quad \therefore OE = \frac{1}{2} AB = 1, \quad \therefore A(-1, 1), \quad B(1, 1),$$

把 $(1, 1)$ 代入 $y = ax^2$ 得, $a = 1$, $\therefore$ 抛物线的解析式为 $y = x^2$;

(2) $x_A \cdot x_B = -1$ 为常数. 理由如下:

如图 2, 过点 A 作 $AM \perp x$ 轴于 M , 过点 B 作 $BN \perp x$ 轴于点 N ,

$$\therefore \angle AMO = \angle BNO = 90^\circ,$$

$$\therefore \angle MAO + \angle AOM = \angle AOM + \angle BON = 90^\circ,$$

$$\therefore \angle MAO = \angle BON, \quad \therefore \triangle AMO \sim \triangle ONB, \quad \therefore \frac{AM}{ON} = \frac{OM}{BN},$$

$$\therefore OM \cdot ON = AM \cdot BN,$$

$$\text{设 } A(x_A, y_A), \quad B(x_B, y_B),$$

$$\therefore A(x_A, y_A), \quad B(x_B, y_B) \text{ 在 } y = x^2 \text{ 的图象上},$$

$$\therefore y_A = x_A^2, \quad y_B = x_B^2, \quad \therefore -x_A \cdot x_B = y_A \cdot y_B = x_A^2 \cdot x_B^2, \quad \therefore x_A \cdot x_B = -1;$$

(3) 由 (2) 知 $x_A \cdot x_B = -1$, 设直线 AB 的解析式为 $y = kx + b$,

$$\text{联立 } \begin{cases} y = kx + b \\ y = x^2 \end{cases}, \text{ 得 } x^2 - kx - b = 0,$$

$\therefore x_A, x_B$ 是该方程的两个根,

$$\therefore x_A + x_B = k, \quad \text{即 } x_B - k = x_A,$$

过点 A 作 $AM \perp x$ 轴于点 M , 过点 B 作 $BN \perp x$ 轴于点 N , 如图 3,

$$\text{由 (2) 可知, } \triangle AMO \sim \triangle ONB, \quad \therefore \frac{OB}{OA} = \frac{BN}{OM},$$

$$\therefore \text{点 } E \text{ 的坐标为 } (k, 0), \quad \therefore OE = -k,$$

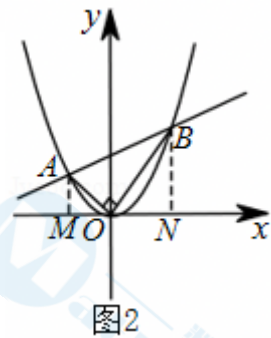
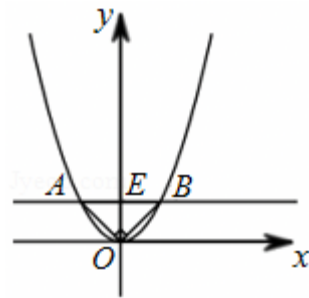


图2

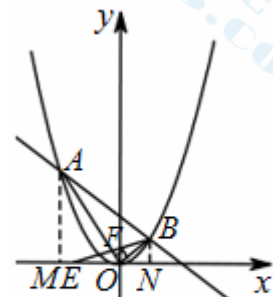


图3

$$\therefore EN = ON + OE = x_B - k = -x_A = OM, \therefore EN = OM, \therefore \frac{OB}{OA} = \frac{BN}{EN},$$

$$\therefore \angle AOB = \angle ENB = 90^\circ,$$

$$\therefore \triangle AOB \sim \triangle ENB,$$

$$\therefore \angle OAB = \angle ENB,$$

$$\therefore \angle AFB = \angle EFO,$$

$$\therefore \triangle EFO \sim \triangle ABF.$$

9. 【解析】(1) $\because$ 抛物线 $y = ax^2 + bx + c$ 与 x 轴交于 $A(-1, 0)$, 与 y 轴交于点 $C(0, -3)$, 对称轴是直线 $x = 1$.

$$\therefore -\frac{b}{2a} = 1, \text{ 解得: } b = -2a,$$

$$\therefore y = ax^2 + bx + c = ax^2 - 2ax + c,$$

将点 A , 点 C 的坐标分别代入 $y = ax^2 - 2ax + c$, 得:

$$\begin{cases} a + 2a + c = 0 \\ c = -3 \end{cases}, \text{ 解得 } \begin{cases} a = 1 \\ c = -3 \end{cases},$$

$$\therefore \text{ 抛物线的解析式为 } y = x^2 - 2x - 3;$$

$$(2) \text{ 解: } D \text{ 点的坐标为 } D_1(-\frac{3}{2}, \frac{9}{4}) \text{ 或 } D_2(\frac{3}{2}, -\frac{15}{4}) \text{ 或 } D_3(1, -4);$$

$\because$ 抛物线 $y = x^2 - 2x - 3$ 与 x 轴交于 $A(-1, 0)$, B 两点, 对称轴为直线 $x = 1$,

$$\therefore B(3, 0),$$

$$\therefore OB = 3, AB = 3 - (-1) = 4,$$

$$\therefore C(0, -3),$$

$$\therefore OC = OB = 3,$$

$$\therefore \angle OCB = \angle OBC,$$

在直角三角形 BOC 中, 由勾股定理得: $BC = \sqrt{3^2 + 3^2} = 3\sqrt{2}$,

①当 $\angle DBC = \angle BAC$ 时, 如图 1.1,

设 BD 交 y 轴于 E , 则 $\triangle BAC \sim \triangle CBE$,

$$\therefore \frac{BA}{CB} = \frac{BC}{CE}, \therefore \frac{4}{3\sqrt{2}} = \frac{3\sqrt{2}}{CE}, \text{ 解得: } CE = \frac{9}{2},$$

$$\therefore OE = CE - OC = \frac{3}{2}, \therefore E(0, \frac{3}{2}),$$

设直线 BD 的解析式为 $y = kx + b$, 将点 B , 点 E 的坐标分别代入得:

$$\begin{cases} 3k + b = 0 \\ b = \frac{3}{2} \end{cases}, \text{ 解得: } \begin{cases} k = -\frac{1}{2} \\ b = \frac{3}{2} \end{cases}, \therefore \text{ 直线 } BD \text{ 的解析式为 } y = -\frac{1}{2}x + \frac{3}{2},$$

$$\text{联立得: } \begin{cases} y = -\frac{1}{2}x + \frac{3}{2} \\ y = x^2 - 2x - 3 \end{cases}$$

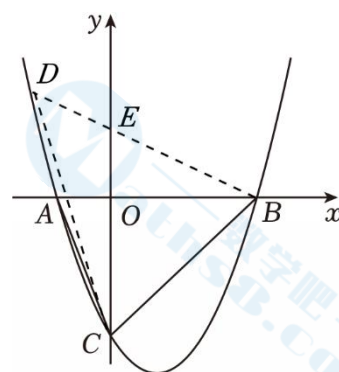


图1.1

解得: $\begin{cases} x = -\frac{3}{2} \\ y = \frac{9}{4} \end{cases}$ 或 $\begin{cases} x = 3 \\ y = 0 \end{cases}$ (不合题意, 舍去), $\therefore D(-\frac{3}{2}, \frac{9}{4})$;

②当 $\angle BCD = \angle BAC$ 时, 作 $BE \perp BC$ 交 CD 的延长线于 E , 作 $EF \perp x$ 轴于点 F , 如图 1.2,

得 $\angle CBA + \angle EBF = 90^\circ$,

$\therefore \angle CBO = 45^\circ$, $\therefore \angle BEF = 45^\circ$,

$$\therefore BF = EF = \frac{\sqrt{2}}{2} BE,$$

由 $\tan \angle BCD = \tan \angle BAC = 3 = \frac{BE}{BC}$,

$$\therefore BE = 3BC = 9\sqrt{2}, \therefore BF = EF = \frac{\sqrt{2}}{2} BE,$$

$$\therefore OF = OB + BF = 3 + 9 = 12,$$

$$\therefore E(12, -9),$$

设直线 CD 的解析式为 $y = ax + b$, 将点 C , 点 E 的坐标

$$\begin{cases} b = -3 \\ 12a + b = -9 \end{cases}, \text{解得} \begin{cases} a = -\frac{1}{2} \\ b = -3 \end{cases}, \therefore \text{直线 } CD \text{ 的解析式为 } y = -\frac{1}{2}x - 3,$$

联立得: $\begin{cases} y = -\frac{1}{2}x - 3 \\ y = x^2 - 2x - 3 \end{cases}$, 解得: $\begin{cases} x = 0 \\ y = -3 \end{cases}$ (不合题意, 舍去) 或 $\begin{cases} x = \frac{3}{2} \\ y = -\frac{15}{4} \end{cases}$,

$$\therefore D(\frac{3}{2}, -\frac{15}{4});$$

③当 $\angle BDC = \angle BAC$ 时, 作 $CE \perp CD$ 交 DB 的延长线于 E , 作 $EM \perp y$ 轴于 M , 如图 1.3,

设 $D(m, m^2 - 2m - 3)$,

由 $\tan \angle BDC = \tan \angle BAC = 3$ 得: $E(-3m^2 + 6m, 3m - 3)$,

设直线 BD 的解析式为 $y = px + q$, 将点 B , 点 D 的坐标分别代入得:

$$\begin{cases} 3p + q = 0 \\ mp + q = m^2 - 2m - 3 \end{cases}, \text{解得: } \begin{cases} p = m + 1 \\ q = -3m - 3 \end{cases},$$

$\therefore$ 直线 BD 的解析式为 $y = (m + 1)x - 3m - 3$, 将点 E 的坐标代入得:

$$(m + 1)(-3m^2 + 6m) - 3m - 3 = 3m - 3,$$

解得: $m_1 = 0$ (不合题意, 舍去), $m_2 = 1$, $\therefore D(1, -4)$,

综上所述, D 点的坐标为 $D_1(-\frac{3}{2}, \frac{9}{4})$ 或 $D_2(\frac{3}{2}, -\frac{15}{4})$ 或 $D_3(1, -4)$;

(3) 证明: 设 $M(m, m^2 - 2m - 3)$, $N(n, n^2 - 2n - 3)$,

联立得: $\begin{cases} y = kx - 2k + 1 \\ y = x^2 - 2x - 3 \end{cases}$, 整理得: $x^2 - (2 + k)x + 2k - 4 = 0$,

$$\therefore m + n = 2 + k, \quad mn = 2k - 4,$$

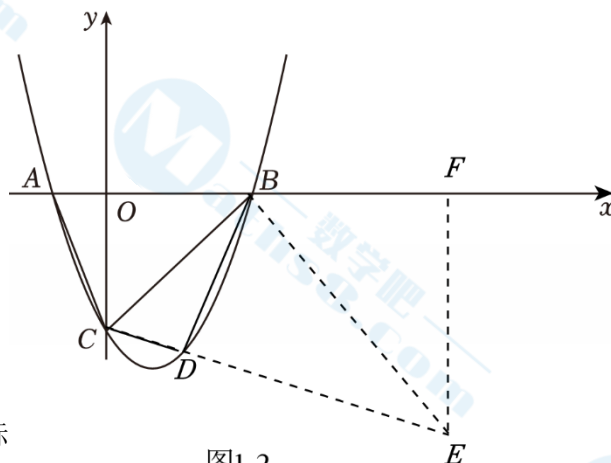


图1.2

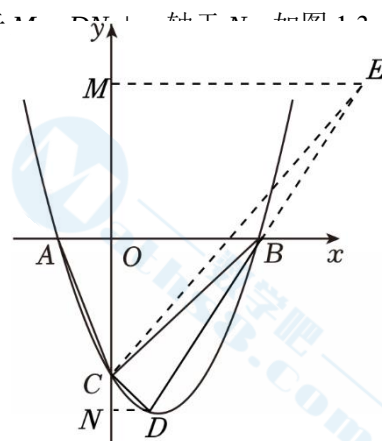


图1.3

消去 k 得 $2(m+n)-mn=8$,

设直线 PN 的解析式为 $y=k_1x+b_1$, 直线 PM 的解析式为 $y=k_2x+b_2$,

联立得: $\begin{cases} y=k_1x+b_1 \\ y=x^2-2x-3 \end{cases}$, 整理得: $x^2-(2+k_1)x-b_1-3=0$,

$\therefore$ 直线与抛物线只有唯一的公共点, $\therefore 2n=2+k_1$, $n^2=-b_1-3$,

$\therefore$ 直线 PN 的解析式为 $y=(2n-2)x-n^2-3$,

同理, 可得直线 PM 的解析式为 $y=(2m-2)x-m^2-3$,

联立得: $\begin{cases} y=(2m-2)x-m^2-3 \\ y=(2n-2)x-n^2-3 \end{cases}$, 解得: 得 $x_p=\frac{m+n}{2}$,

$\therefore y_p=mn-m-n-3$,

又 $\because 2(m+n)-mn=8$,

$\therefore y_p=m+n-11=2x_p-11$,

$\therefore$ 点 P 在定直线 $y=2x-11$.

10. 【解析】(1) $\because$ 一次函数 $y=\frac{5}{4}x+m$ (m 为常数) 的图象与 x 轴交于点 $A(-3,0)$,

$\therefore 0=\frac{5}{4} \times (-3)+m$, 解得 $m=\frac{15}{4}$,

$\therefore$ 一次函数解析式为 $y=\frac{5}{4}x+\frac{15}{4}$, $\therefore C$ 点坐标为 $(0, \frac{15}{4})$.

$\therefore$ 以直线 $x=1$ 为对称轴的抛物线 $y=ax^2+bx+c$ (a 、 b 、 c 为常数, 且 $a \neq 0$) 经过 $A(-3,0)$ 、 $C(0, \frac{15}{4})$,

$\therefore \begin{cases} -\frac{b}{2a}=1 \\ 9a-3b+c=0 \\ c=\frac{15}{4} \end{cases}$, 解得 $\begin{cases} a=-\frac{1}{4} \\ b=\frac{1}{2} \\ c=\frac{15}{4} \end{cases}$, $\therefore$ 抛物线的函数表达式为 $y=-\frac{1}{4}x^2+\frac{1}{2}x+\frac{15}{4}$;

(2) 存在. 设 $Q(x, -\frac{1}{4}x^2+\frac{1}{2}x+\frac{15}{4})$.

①当点 C 为直角顶点时, 如图, 作 $CQ \perp AC$ 交抛物线于点 Q , $QE \perp y$ 轴于 E .

在 $\triangle ACO$ 与 $\triangle CQE$ 中,

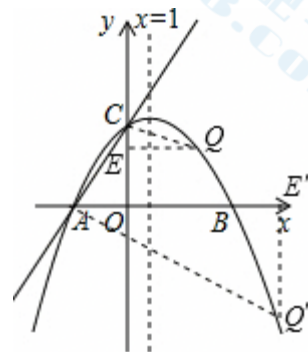
$\begin{cases} \angle ACO = \angle CQE = 90^\circ - \angle QCE \\ \angle AOC = \angle CEQ \end{cases}$, $\therefore \triangle ACO \sim \triangle CQE$,

$\therefore \frac{QE}{CO} = \frac{CE}{AO}$, 即 $\frac{x}{\frac{15}{4}} = \frac{\frac{15}{4} - (-\frac{1}{4}x^2 + \frac{1}{2}x + \frac{15}{4})}{3}$,

解得 $x_1=5.2$, $x_2=0$ (不合题意舍去);

②当点 A 为直角顶点时, 如图, 作 $AQ' \perp AC$ 交抛物线于点 Q' , $Q'E' \perp x$ 轴于 E' .

在 $\triangle ACO$ 与 $\triangle Q'AE'$ 中,



$$\begin{cases} \angle OAC = \angle E'Q'A = 90^\circ - \angle OAQ', & \therefore \triangle ACO \sim \triangle Q'AE', \\ \angle AOC = \angle Q'E'A \end{cases}$$

$$\therefore \frac{AE'}{CO} = \frac{Q'E'}{AO}, \text{ 即 } \frac{\frac{x+3}{15}}{\frac{15}{4}} = \frac{\frac{1}{4}x^2 - \frac{1}{2}x - \frac{15}{4}}{3},$$

解得 $x_1 = 8.2$, $x_2 = -3$ (不合题意舍去)。

综上所述: Q 点的横坐标为 5.2 或 8.2;

$$(3) \because y = -\frac{1}{4}x^2 + \frac{1}{2}x + \frac{15}{4} \text{ 与 } x \text{ 轴交于 } A(-3, 0)、B \text{ 两点, 对称轴为直线 } x = 1, \therefore B \text{ 点坐标为 } (5, 0),$$

$$\therefore C(0, \frac{15}{4}), \therefore \text{直线 } BC \text{ 的解析式为 } y = -\frac{3}{4}x + \frac{15}{4},$$

$$\text{当 } x = 1 \text{ 时, } y = -\frac{3}{4} \times 1 + \frac{15}{4} = 3, \therefore P(1, 3).$$

设过点 P 的直线为: $y = kx + 3 - k$,

$$\text{把 } y = kx + 3 - k \text{ 代入 } y = -\frac{1}{4}x^2 + \frac{1}{2}x + \frac{15}{4},$$

$$\text{得 } kx + 3 - k = -\frac{1}{4}x^2 + \frac{1}{2}x + \frac{15}{4},$$

$$\text{整理得, } x^2 + (4k - 2)x - 4k - 3 = 0,$$

$$\therefore x_1 + x_2 = 2 - 4k, \quad x_1 x_2 = -4k - 3, \quad y_1 - y_2 = k(x_1 - x_2),$$

$$\therefore (x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1 x_2 = (2 - 4k)^2 - 4(-4k - 3) = 16k^2 + 16,$$

$$\therefore M_1 M_2 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{1 + k^2} \sqrt{(x_1 - x_2)^2} = 4(1 + k^2),$$

$$\text{同理: } M_1 P = \sqrt{(x_1 - 1)^2 + (kx_1 + 3 - k - 3)^2} = \sqrt{1 + k^2} \sqrt{(x_1 - 1)^2},$$

$$M_2 P = \sqrt{1 + k^2} \sqrt{(x_2 - 1)^2},$$

$$\therefore M_1 P \cdot M_2 P = \sqrt{1 + k^2} \sqrt{(x_1 - 1)^2} \cdot \sqrt{1 + k^2} \sqrt{(x_2 - 1)^2} = |(x_1 - 1)(x_2 - 1)| \cdot (1 + k^2) = 4(1 + k^2),$$

$$\therefore \frac{M_1 P \cdot M_2 P}{M_1 M_2} = 1 \text{ 为定值.}$$